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Transmitting qudits through larger quantum channels

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Abstract
We address the problem of transmitting states belonging to finite dimensional Hilbert space through a quantum channel associated with a larger (even infinite dimensional) Hilbert space.

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1. Introduction

By a quantum channel is intended [1, 2] a completely positive and trace-preserving linear map $\Phi : \sigma(H) \to \sigma(H)$, where $\sigma(H)$ is a set of states in a Hilbert space H with $\dim H = n \leqslant +\infty$. Even for a finite dimension $n < +\infty$ there is a number of difficult problems concerning the construction of optimal transmission of the information through a quantum channel. The infinite dimensional case we are especially interested in has additional particular features [3]. In the present paper, we discuss how a set of states on the finite dimensional Hilbert space can be transmitted through a quantum channel associated with the infinite dimensional Hilbert space. In particular, we investigate whether encoding a qudit in a larger space could be useful to better protect it from the channel (noise) action without resorting to any particular decoding (recovery) scheme at the output.

The paper is organized as follows. In section 2 the notion of subchannel is introduced. In section 3 we study the phase and the amplitude damping channels within this context. Section 4 is for conclusions.

2. Invariant hulls and subchannels

Given a subspace $K \subset H$, $\dim K = d < n$, we denote by $\sigma(K)$ the convex envelope of pure states $|\xi\rangle\langle\xi| \subset \sigma(H)$, $\xi \in K$. One can define a linear map $\Psi : \sigma(K) \to \sigma(K)$ by the formula

$$\mathrm{Tr}(x_2\Psi(x_1)) = \mathrm{Tr}(x_2\Phi(x_1)), \qquad x_i \in \sigma(K), \quad i = 1, 2. \tag{1}$$

Substituting $x_1 = |\xi\rangle\langle\xi|$ into equation (1) we obtain $\langle\xi|\Psi(x_2)|\xi\rangle = \langle\xi|\Phi(x_2)|\xi\rangle$. It follows that $\Psi(x) = P_K\Phi(x)P_K$, where P_K is a projection on the subspace K and $x \in \sigma(K)$. Hence Ψ is a completely positive map. If Φ is unital, i.e. it preserves the chaotic state $\Phi\left(\frac{1}{n}I\right) = \frac{1}{n}I$ (with I the identity in H), then Ψ satisfies the property $\Psi\left(\frac{1}{d}P_K\right) = \frac{1}{d}P_K$.

Consider the set of states

$$\mathrm{Im_K}\Phi \equiv \{\mathrm{x} \in \sigma(\mathrm{H}) \mid \exists \mathrm{y} \in \sigma(\mathrm{K}) : \Phi(\mathrm{y}) = \mathrm{x}\}. \tag{2}$$

If $\mathrm{Im_K}\Phi \subset \sigma(\mathrm{K})$, then we shall call $\sigma(K)$ *an invariant hull* of the channel Φ. In that case we get $\Psi = \Phi|_{\sigma(K)}$, where $\Phi|_{\sigma(K)}$ stands for the restriction of Φ to inputs in $\sigma(K)$. Because it implies that Ψ is trace-preserving, we shall call Ψ *a subchannel* of Φ.

We denote by $B(H)$ the algebra of all bounded operators in H. Due to the Kraus decomposition[4] for the channel Φ there exist a set of operators $E = \{E_i \in B(H), 1 \leqslant i \leqslant k \leqslant n^2\}$ such that

$$\Phi(x) = \sum_{i=1}^{k} E_i x E_i^*, \qquad x \in \sigma(H). \tag{3}$$

where E_i^* stands for the adjoint of E_i.

Example 1. Consider the phase damping channel defined with $n = +\infty$ through the decomposition

$$\Phi(x) = \sum_{i=0}^{+\infty} E_i x E_i^*, \tag{4}$$

where

$$E_i = \sum_{k=0}^{\infty} \frac{[k\sqrt{-2\ln\eta}]^i}{\sqrt{i!}}[\eta]^{k^2}|k\rangle\langle k|. \tag{5}$$

Here $|k\rangle, k = 0, 1, 2, \ldots,$ are Fock states and the parameter η describes the damping (it can be written as $\eta = \mathrm{e}^{-\gamma t}$ with γ being the damping rate and t the transmission time). The property $\Phi(|k\rangle\langle k|) = |k\rangle\langle k|$ guarantees that any subspace K generated by the vectors $|k_0\rangle, \ldots, |k_{d-1}\rangle$ determines the invariant hull of the phase damping channel.

Example 2. Consider the amplitude damping channel defined with $n = +\infty$ through the decomposition

$$\Phi(x) = \sum_{i=0}^{+\infty} E_i x E_i^*, \tag{6}$$

where

$$E_i = \sum_{k=i}^{+\infty} \sqrt{C_k^i}[\eta]^{(k-i)/2}[1-\eta]^{i/2}|k-i\rangle\langle k|, \tag{7}$$

with

$$C_k^i = \frac{k!}{(k-i)!i!} \tag{8}$$

Here again $|k\rangle, k = 0, 1, 2, \ldots,$ are Fock states and the parameter η describes the damping. For a given dimension d there exists the invariant hull for Φ. In fact, if K is generated by a collection of vectors $|0\rangle, \ldots, |d-1\rangle$, then $\sigma(K)$ is an invariant hull for Φ.

[4] This terminology arose because of Kraus' book [4] where the decomposition appeared; however, it was first proposed in [5].

Example 3. Consider the depolarizing channel defined as $\Phi(x) = px + (1-p)\frac{1}{n}I$, with $0 \leqslant p \leqslant 1$. Note that if $K \neq H$, then the chaotic state $\frac{1}{n}I \notin \sigma(K)$. Hence the depolarizing channel has no invariant hull if $d < n$ because $\mathrm{Im_K}\Phi \notin \sigma(\mathrm{K})$ under this condition.

For any channel Φ in the Hilbert space H one can define the conjugate unital completely positive map Φ^* as follows:

$$\mathrm{Tr}(x_1\Phi^*(x_2)) = \mathrm{Tr}(\Phi(x_1)x_2), \qquad x_1 \in \sigma(H), \quad x_2 \in B(H). \tag{9}$$

Because Φ is trace-preserving, we obtain that Φ^* is unital. Moreover, to check that the map Φ is trace-preserving it is sufficient to look whether Φ^* is unital or not. Equation (3) allows us to extend Φ from $\sigma(H)$ to $B(H)$. For the conjugate map Φ^* we get

$$\Phi^*(x) = \sum_{i=1}^{k} E_i^* x E_i, \qquad x \in B(H). \tag{10}$$

Now let $\Psi(x) = P_K\Phi(x)P_K, x \in \sigma(K)$. The map Ψ is a subchannel of Φ iff it is trace-preserving. To fulfil this property it needs that Ψ^* is unital in the sense $\Psi^*(P_K) = P_K$. On the other hand, it takes place iff

$$P_K\Phi^*(P_K)P_K = P_K. \tag{11}$$

Hence, the problem of searching for subchannels of Φ is equivalent to the problem of describing the algebra of fixed elements for the map $P_K\Phi^*(\cdot)P_K$.

If Ψ is a subchannel of Φ and $\dim K = 2$, then we shall say that Ψ is a *qubit subchannel* of Φ. Let

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|\psi_0\rangle + \mathrm{e}^{\mathrm{i}\phi}\sin\left(\frac{\theta}{2}\right)|\psi_1\rangle, \qquad \rho = |\psi\rangle\langle\psi|, \tag{12}$$

with $|\psi_0\rangle, |\psi_1\rangle$ spanning K. A way to see how faithfully the state ρ is transmitted through the channel Φ is to consider the fidelity distance [2]

$$f(\theta,\phi) = \mathrm{Tr}(\rho, \Phi(\rho)) \equiv \mathrm{Tr}\left\{\sqrt{\sqrt{\rho}\Phi(\rho)\sqrt{\rho}}\right\} = \langle\psi|\Phi(\rho)|\psi\rangle, \tag{13}$$

and then average overall the Bloch sphere, to get

$$\mathcal{F} = \frac{1}{4\pi}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\theta\,\sin(\theta)f(\theta,\phi). \tag{14}$$

It turns out that the fidelity of the channel Φ is equal to the fidelity of its qubit subchannel Ψ (with the subspace K generated by the vectors $|\psi_0\rangle$ and $|\psi_1\rangle$). Following this way, we can conclude that to estimate how well a channel Φ preserves a qubit state, one should consider all qubit subchannels of Φ.

3. Applications

3.1. The phase damping channel

Suppose that Φ represents the phase damping channel defined in example 1 of section 2. The projectors $|k\rangle\langle k|, k = 0, 1, 2, \ldots,$ belong to the algebra of fixed elements of Φ. Hence, any subspace K, being a linear envelope of the vectors $|k\rangle$ and $|s\rangle$, $k \neq s$, generates the unital qubit subchannel of Φ. Note that as a consequence of equations (4) and (5) we have

$$\Phi(|k\rangle\langle s|) = \eta^{(k-s)^2}|k\rangle\langle s|. \tag{15}$$

Thus, by referring to equation (12) with $|k\rangle \equiv |\psi_0\rangle$ and $|s\rangle \equiv |\psi_1\rangle$, we get

$$\mathrm{Tr}(\rho, \Phi(\rho)) = \cos^4\left(\frac{\theta}{2}\right) + \sin^4\left(\frac{\theta}{2}\right) + 2\eta^{(k-s)^2}\cos^2\left(\frac{\theta}{2}\right)\sin^2\left(\frac{\theta}{2}\right); \tag{16}$$

hence the fidelity

$$\mathcal{F} = \frac{2}{3} + \frac{\eta^{(k-s)^2}}{3}. \tag{17}$$

The maximum is achieved for k, s contiguous natural numbers.

3.2. The amplitude damping channel

Suppose that Φ represents the amplitude damping channel defined in example 2 of section 2. Take two integer numbers $0 \leqslant k \leqslant s$; then as a consequence of equations (6)–(8), we have

$$\Phi(|k\rangle\langle s|) = \sum_{i=0}^{k} \sqrt{C_k^i C_s^i}\,\eta^{\frac{k+s}{2}-i}(1-\eta)^i |k-i\rangle\langle s-i|. \tag{18}$$

If $x = \sum_{k=0}^{+\infty} x_{kl}|k\rangle\langle l|$, then for $y = \Phi(x) = \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty} y_{kl}|k\rangle\langle l|$ we obtain

$$y_{kl} = \sum_{i=0}^{+\infty} \sqrt{C_{k+i}^i C_{l+i}^i}\,\eta^{\frac{k+l}{2}}(1-\eta)^i x_{k+il+i}. \tag{19}$$

It follows from equation (19) that $\Phi(x) = x$ iff $x = \mathrm{const}|0\rangle\langle 0|$. Hence, there is no unital qubit subchannel of the amplitude damping channel. In fact, if the subspace K generates an invariant qubit of Φ, then the subchannel $\Psi = \Phi|_{\sigma(K)}$ is unital only if $\Phi(P_K) = P_K$, where P_K is a two-dimensional projection on K.

Example 4. Given a complex number $\alpha \in \mathbb{C}$ one can define the coherent state by the formula

$$|\alpha\rangle = \mathrm{e}^{-\frac{|\alpha|^2}{2}} \sum_{k=0}^{+\infty} \frac{\alpha^k}{\sqrt{k!}}|k\rangle. \tag{20}$$

It follows from equations (18) and (20) that, for any $\alpha, \beta \in \mathbb{C}$, we get

$$\Phi(|\alpha\rangle\langle\beta|) = |\sqrt{\eta}\alpha\rangle\langle\sqrt{\eta}\beta| \exp\left[(1-\eta)\left(-\frac{|\alpha|^2+|\beta|^2}{2} + \alpha\beta^*\right)\right]. \tag{21}$$

In particular,

$$\Phi(|\alpha\rangle\langle\alpha|) = |\sqrt{\eta}\alpha\rangle\langle\sqrt{\eta}\alpha| \tag{22}$$

for any coherent state $|\alpha\rangle\langle\alpha|$. Equation (22) implies that the Schrödinger cat states $|\psi_\pm\rangle = \mathcal{N}_\pm(|\alpha\rangle \pm |-\alpha\rangle)$ do not form an invariant qubit for Φ.

Quite generally we can consider the qubit subchannel Ψ of the amplitude damping Φ generated by the vectors $|\psi_0\rangle, |\psi_1\rangle$ (basis for the qubit subspace K) given by

$$|\psi_0\rangle = \sum_{n=0}^{+\infty} c_n|n\rangle, \qquad |\psi_1\rangle = \sum_{n=0}^{+\infty} d_n|n\rangle. \tag{23}$$

The conditions

$$\sum_{n=0}^{+\infty} |c_n|^2 = 1, \qquad \sum_{n=0}^{+\infty} |d_n|^2 = 1 \tag{24}$$

and

$$\sum_{n=0}^{+\infty} \overline{c}_n d_n = 0 \tag{25}$$

ensure the normalization of $|\psi_0\rangle$ and $|\psi_1\rangle$ and their orthogonality. Then, we consider a generic qubit state as in equation (12). Note that the completely positive map $\Psi = \Phi|_{\sigma(K)}$ is a non-unital qubit subchannel of Φ at least if $c_0 = 1$ and $d_1 = 1$.

From equation (14) we get the fidelity as

$$\mathcal{F} = \frac{1}{6} \sum_{k=0}^{+\infty} \sum_{n,m=k}^{+\infty} \sqrt{C_n^k C_m^k} [\eta]^{(n+m-2k)/2} [1-\eta]^{k/2} [c_n c_m (d_{m-k} \overline{d}_{n-k} + 2c_{m-k}\overline{c}_{n-k}) \\ + d_n \overline{d}_m (c_{m-k}\overline{c}_{n-k} + 2d_{m-k}\overline{d}_{n-k}) + d_n \overline{d}_{n-k} \overline{c}_m c_{m-k} + c_n \overline{c}_{n-k} \overline{d}_m d_{m-k}]. \tag{26}$$

Now we should maximize the fidelity overall possible choices of $\{c_n\}$ and $\{d_n\}$. Clearly, there are no two simultaneous and orthogonal eigenstates of Ψ; hence the maximum of $\mathcal{F}$ cannot be 1.

If we use only the first two vectors $|0\rangle$ and $|1\rangle$ to parametrize the qubit, then any choice of c_0, c_1, d_0, d_1 obeying the orthogonality condition (25) gives us a correctly defined non-unital qubit subchannel $\Psi = \Phi|_{\sigma(K)}$. In that case, the fidelity results,

$$\mathcal{F} = \frac{1}{2} + \frac{\eta}{6} + \frac{\sqrt{\eta}}{3}, \tag{27}$$

and it does not depend on the choice of c_0, c_1, d_0, d_1.

One step ahead is to parametrize the qubit by the first three vectors $|0\rangle, |1\rangle, |2\rangle$ as

$$|\psi_0\rangle = \sin\alpha \cos\beta|0\rangle + \sin\alpha \sin\beta|1\rangle + \cos\alpha|2\rangle, \tag{28}$$

$$|\psi_1\rangle = \sin\gamma \cos\delta|0\rangle + \sin\gamma \sin\delta|1\rangle + \cos\gamma|2\rangle, \tag{29}$$

with the condition

$$\sin\alpha \cos\beta \sin\gamma \cos\delta + \sin\alpha \sin\beta \sin\gamma \sin\delta + \cos\alpha \cos\gamma = 0. \tag{30}$$

Note that we have skipped all relative phases because of the symmetry of amplitude damping channel action. So, in practice, we only deal with three free parameters, α, β, γ. In that case, the fidelity is upper bounded by equation (27) and such a bound is achieved with $\alpha = \gamma = \frac{\pi}{2}$ and independently of β. The same takes place when parametrizing the qubit with more than three states.

Thus, it clearly results that the fidelity is optimized by encoding the qubit into the lowest two Fock states of the Hilbert space (namely $|0\rangle$ and $|1\rangle$). This is in agreement with the physical arguments based on energy (amplitude) damping. We conjecture that such a result can be extended from qubit to qudit subchannels. That is, the optimal (in terms of fidelity) way to transmit a qudit through an $(n \geqslant d)$-dimensional amplitude damping channel would be to encode it into the lowest (in terms of energy) d orthogonal states $|0\rangle, |1\rangle, \ldots, |d-1\rangle$.

4. Conclusion

We have formally addressed the problem of transmitting qudits through larger quantum channels by introducing the concepts of invariant hulls and subchannels. We have considered qudits encoded in the larger space of the channel without resorting to any particular decoding (recovery) scheme at the output. After applying these arguments to specific examples, it comes out that to send a qubit through an infinite dimensional phase damping channel the best would be to encode it into two contiguous Fock states, while to send a qubit through an infinite dimensional amplitude damping channel the best would be to encode it into the two lowest Fock states. Although these results are derived from an information theoretic approach, they are in agreement with those coming from physical arguments. These results can be generalized to qudits.

Moreover, from the presented examples, it turns out that transmitting qudits in a channel of dimension greater than d (even infinity) does not allow for a better fidelity, with respect to the case of a channel of dimension d. Nevertheless, we believe that the extra space could be profitably exploited with suitable decoding, recovery procedures (see also [6]). From now we may argue that they cannot simply be completely positive trace-preserving maps operating at the output of the channel as these are not able to decrease the distance between input and output states. Some possibilities will be addressed in future works.

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